Deconstructing a pointing model for the 40M OAN radiotelescope

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Informe Técnico IT-OAN 2007-26

Change Record

Version	Date	Author	Remarks
1.0	30-10-2003	P. de Vicente	First version. Never published
2.0	30-11-2007	P. de Vicente	Revised, rewritten and corrected

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1. Introduction

Using a pointing model is a common practice among radiotelescopes devoted to astronomy to correct for mechanical missalignments. The OAN 40m radiotelescope has been designed by MAN Techonologie, who, together with BBH, is also in charge of the servosystem design and implementation. Based on the experience of antennas for communications which lack pointing models, MAN provides a procedure for the 40M missalignment errors which is summarized in figure 1. According to that diagram the astronomical pointing model will be applied after other corrections which take into account gravity, readouts from the encoders and tables for the encoders are applied. This means that probably the astronomical pointing model will correct for very small values since the effects it takes into account will have been corrected previously.

There are two different philosophies to create a pointing model for a telescope. The most extended philosophy consists on using a model that describes real effects on the mount of the telescope, of the mirrors and defects on the encoders. In this case the parameters are directly associated with physical effects. Nevertheless, in some telescopes empirical functions are added to remove residual effects whose physical cause is unknown or poorly understood. The second philosophy just uses a set of empirical functions, like spherical harmonics, unrelated to any physical effect, to correct for the pointing errors.

We believe that a physical model is more desirable since it allows to identify the source of the pointing errors and may help in removing or minimizing the causes.

The algorithm we propose is very similar to that used by ALMA (Mangum 2001) although we will add one more term for gravity as in the IRAM 30M radiotelescope (Greve et al. 1996), and we will also take into account the missalignment of the Nasmyth mirrors (Barcia 2003). The latter produce errors which have the same dependency on azimuth and elevation as some classic ones like the constant offsets for azimuth and elevation, the tilt of the azimuth axis or gravitational effects, and therefore are not distinguishable from them. However this has no effect on the algorithm itself. We will use a similar notation as the one used by Greve et al. (1996) for the 30M, instead of using the one from Magnum (2001) for ALMA.

2. Models in other radiotelescopes: ALMA and the 30M

The models in other telescopes define the pointing errors in azimuth and elevation. That is,

$$\Delta Az = Az_{observed} - Az_{encoder} \tag{1}$$
$$\Delta El = El_{observed} - El_{encoder} \tag{2}$$

where ΔAz and ΔEl are given by the model. We assume that $Az_{encoder}$ and $El_{encoder}$ are equivalent to commanded azimuth and elevation.

A pointing model proposed for ALMA is described by Magnum (2001) in ALMA memo 366. The equations for correcting the azimuth and elevation are:

$$\begin{aligned} \Delta Az &= IA + CA \sec El + NPAE \tan El + AN \tan El \sin Az - AW \tan El \cos Az \\ &\quad + A_{obs} \sec El \qquad (3) \\ \Delta El &= IE + ECEC \cos El + AN \cos Az + AW \sin Az + E_{obs} \qquad (4) \end{aligned}$$

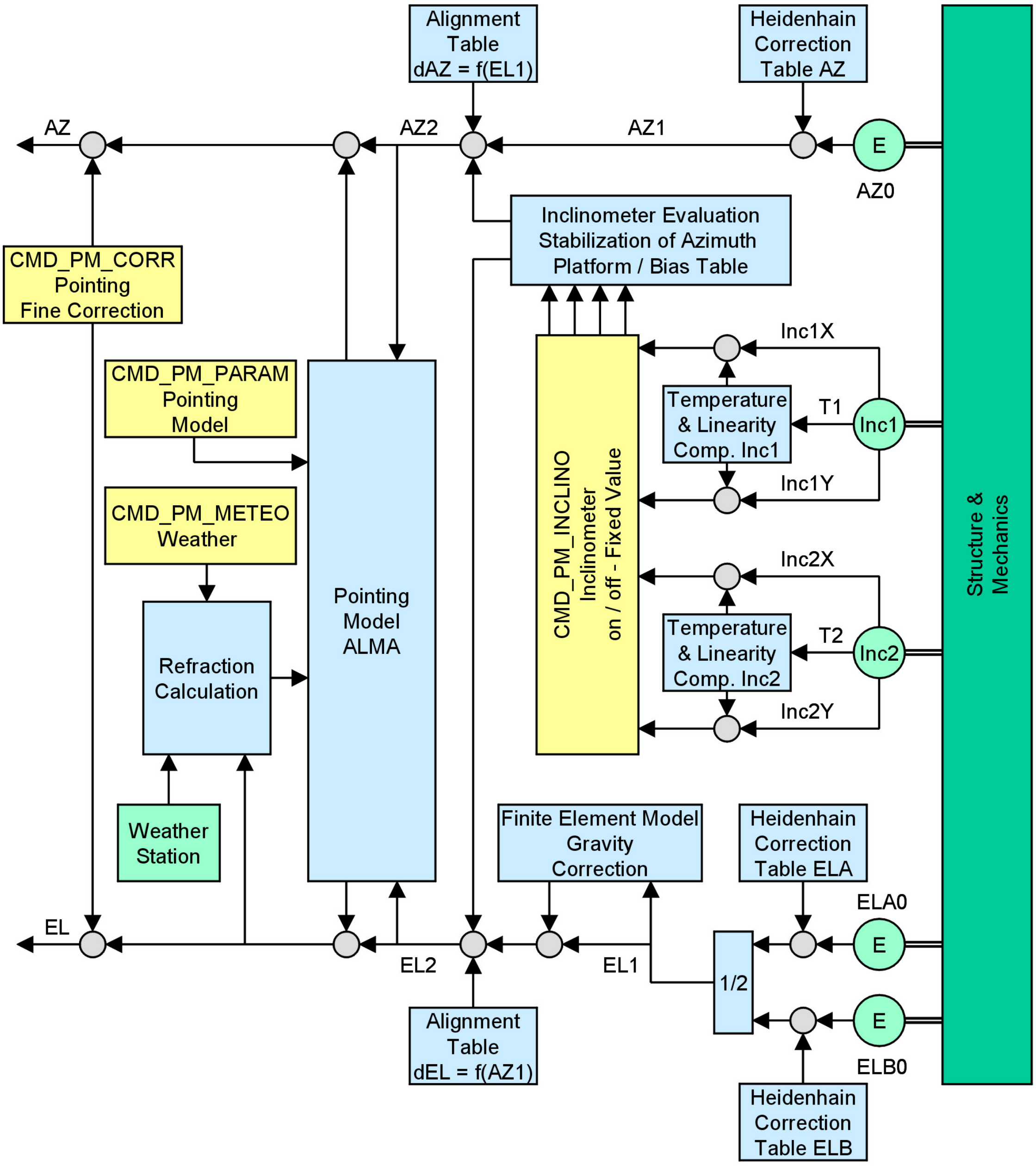

Figura 1: *Pointing correction block diagram proposed by MAN for the 40m dish*

where the individual pointing coefficients are defined in Table 1.

Magnum (2001) also defines secondary additional pointing equations which should be applied for each receiver:

$$\Delta Az_1 = IA_1 + CA_1 \sec El \quad (5)$$

$$\Delta El_1 = IE_1 + ECEC_1 \cos El \quad (6)$$

The pointing model for the IRAM 30m telescope was first described by Greve et al. (1996). The equations for the azimuth and elevation errors are:

$$\Delta Az = P_1 + P_2 \sec El + P_3 \tan El + P_4 \tan El \cos Az + P_5 \tan El \sin Az + P_6 \sin Az \quad (7)$$

$$\Delta El = P_7 - P_4 \sin Az + P_5 \cos Az + P_8 \cos El + P_9 \sin El + P_6 \cos Az \sin El \quad (8)$$

where the individual pointing coefficients are defined in Table 1.

ALMA	IRAM 30M	Physical meaning
IA	P_1	Azimuth encoder offset
CA	P_2	Collimation error, also known as non orthogonality between the radio beam and the elevation axis
$NAPE$	P_3	Lack of orthogonality between the azimuth and elevation exis
AN	P_4	Tilt of azimuth axis along a N-S direction
AW	P_5	Tilt of azimuth axis along a E-W direction
A_{obs}		Observer applied azimuth correction
	P_6	Declination error of the source
IE	P_7	Elevation encoder offset
$ECEC$	P_8	Gravitational flexure
	P_9	Gravitational bending
E_{obs}		Observer applied azimuth correction

Cuadro 1: *Pointing coefficients for ALMA and the IRAM 30m telescope*

As can be seen from equations 3, 4, 7 and 8 there are some sign discrepancies between both models and there are also coefficients which only appear in one model. In the following sections we will understand the reason for these differences and why we believe these discrepancies are not important.

3. The coordinate system

We will represent the radiotelescope pointing position (azimuth and elevation) in a cartesian coordinate system where the origin is placed at the radiotelescope. Usually it will be where the azimuth and the elevation axis cross, but since this may not happen we will consider the place where the azimuth axis and a plane containing the elevation axis and parallel to the ground cross. The X axis lays along a West-East line, where values increase from West to East. The Y axis lays along a South-North line, where values increase from South to North. The Z axis lays along a line from the Nadir to the Zenith, where values increase towards the Zenith.

The azimuth (Az) is the angle between the direction of pointing of the telescope and the YZ plane, and is measured from the North clockwise. The elevation (El) is the angle between the direction of pointing and the XY plane. Figure 2 shows graphically these definitions.

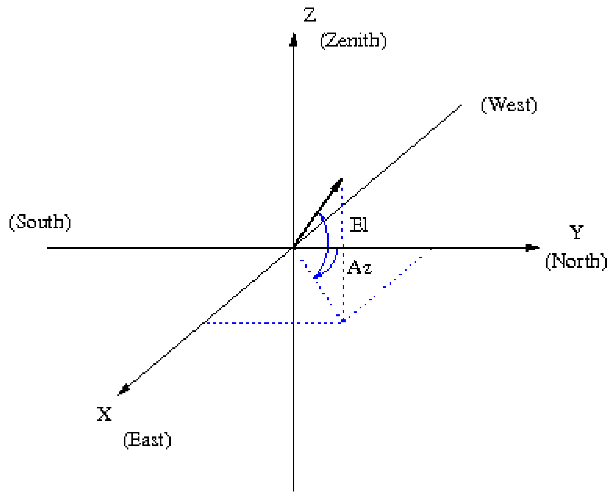

Figura 2: *System of coordinates. Definition for azimuth and elevation*

Let us assume that $\vec{p}$ is a unitary vector along the direction of pointing of the antenna. It can be expressed as:

$$\vec{p} = (i, j, k)$$

where i, j and k are the components of $\vec{p}$ along the X, Y and Z axis respectively.

The pointing vector can also be expresed in terms of azimuth and elevation as follows:

$$\vec{p} = (i, j, k) = (\sin Az \cos El, \cos Az \cos El, \sin El) \quad (9)$$

$$= (\cos(\pi/2 - Az) \cos El, \sin(\pi/2 - Az) \cos El, \sin El) \quad (10)$$

It is important to highlight again that angle Az is measured from the Y axis clockwise and usually when representing coordinates in polar coordinates the angle is measured from the X axis counterclockwise. The azimuth is then:

$$Az = \arctan \frac{i}{j} \quad (11)$$

and the elevation is:

$$El = \arcsin k \quad (12)$$

4. Remembering how to rotate around the axis

The telescope pointing errors will be obtained making a coordinate transformation between the current system and one where the errors are null. Any such transformation can be decomposed in a consecutive sum of rotations around the X, Y and Z axis. In most of the cases the rotations will be small and the difference between the angles of azimuth and elevation in one system and in the other will give the error in azimuth and in elevation.

A clockwise rotation by an angle α around the X axis is described by:

$$\vec{p}\,' = \begin{pmatrix} i' \\ j' \\ k' \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha \\ 0 & -\sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} i \\ j \\ k \end{pmatrix} \tag{13}$$

A clockwise rotation by an angle β around the Y axis is described by:

$$\vec{p}\,' = \begin{pmatrix} i' \\ j' \\ k' \end{pmatrix} = \begin{pmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{pmatrix} \begin{pmatrix} i \\ j \\ k \end{pmatrix} \tag{14}$$

A clockwise rotation by an angle γ around the Z axis is described by:

$$\vec{p}\,' = \begin{pmatrix} i' \\ j' \\ k' \end{pmatrix} = \begin{pmatrix} \cos\gamma & \sin\gamma & 0 \\ -\sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} i \\ j \\ k \end{pmatrix} \tag{15}$$

5. Deconstructing the model for a cassegrain antenna

To deconstruct the pointing model we will use the nomenclature used by Greve et al (1996). We will analyze each of the terms in the following subsections

5.1. Constant offset for azimuth. $\mathbf{P_1^{enc}}$

This term is due to a constant readout error in the azimuth encoder, which can be observed when commanding the antenna to azimuth 0. In such situation the real position differs from the commanded position by P_1^{enc}. This term causes no error in elevation.

5.2. Constant offset for elevation. $\mathbf{P_7^{enc}}$

This term is due to a constant readout error in the elevation encoder, which can be observed when commanding the antenna to elevation 0. In such situation the real position differs from the commanded position by P_7^{enc}. This term causes no error in azimuth.

5.3. Collimation error. P_2^c and P_7^c

In a paraboloid with a feed in the primary focus, if the feed is out of the mechanical axis, the electrical axis deviates from the mechanical axis by a quantity (angle) proportional to the angle formed by this axis with the line that connects the vertex of the paraboloid and the feed. The constant of proportionality is the "beam deviation factor" and increases with the focal/diameter relationship (f/D) up to a maximum value of 1. In the Cassegrain antennas the effect of the shift of the feed is analyzed treating them as antennas in the primary focus with the equivalent paraboloid focal distance. Since the f/D is very big, the beam deviation factor is 1, and the angle of the electrical axis with the mechanical one is equal to the angle formed by the mechanical axis and the line that connects the vertex of the reflector with the feed.

This deviation may be decomposed in two components:

1. one on the plane perpendicular to the elevation axis that contains the mechanical axis. We will call this P_7^c. P_7^c causes a constant pointing error in elevation, which is given by:

$$\Delta El = P_7^c \tag{16}$$

2. one on the plane that contains the elevation axis and the mechanical axis. This angle will be called P_2^c.

The P_2^c error may also be described as the electrical axis being not perpendicular to the elevation axis, and causes a pointing error in azimuth which only depends on elevation, as we will prove on this section.

We may assume for commodity and without loss of generality, that the current azimuth is 0 (the antenna points towards the North) and its elevation is El. See figure 3. We will assume that the optical axis is not perpendicular to the elevation axis by a positive angle P_2^c.

To move to a coordinate system where the error is 0, we will rotate the system around the X axis by $-El$ so that the direction of pointing is contained in the new XY (ground) plane. See figure 4.

Then we will rotate an angle P_2^c around Z_1, the new Z axis. After this rotation the pointing direction lays along Y_2 as shown in Figure 5.

Finally we undo the first transformation by rotating an angle El around the X_2 axis.

We will call $\vec{p_3}$ the new unitary vector in the direction of pointing. Subscript 3 indicates that we have performed 3 transformations:

$$\vec{p_3} = \begin{pmatrix} i_3 \\ j_3 \\ k_3 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos El & -\sin El \\ 0 & \sin El & \cos El \end{pmatrix} \begin{pmatrix} \cos P_2^c & -\sin P_2^c & 0 \\ \sin P_2^c & \cos P_2^c & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos El & \sin El \\ 0 & -\sin El & \cos El \end{pmatrix} \begin{pmatrix} i \\ j \\ k \end{pmatrix}$$

The complete transformation is described by:

$$\vec{p_3} = \begin{pmatrix} \cos P_2^c & -\cos El \sin P_2^c & -\sin El \sin P_2^c \\ \cos El \sin P_2^c & \cos^2 El \cos P_2^c + \sin^2 El & \sin El \cos El (\cos P_2^c - 1) \\ \sin El \sin P_2^c & \sin El \cos El (\cos P_2^c - 1) & \sin^2 El \cos P_2^c + \cos^2 El \end{pmatrix} \begin{pmatrix} i \\ j \\ k \end{pmatrix}$$

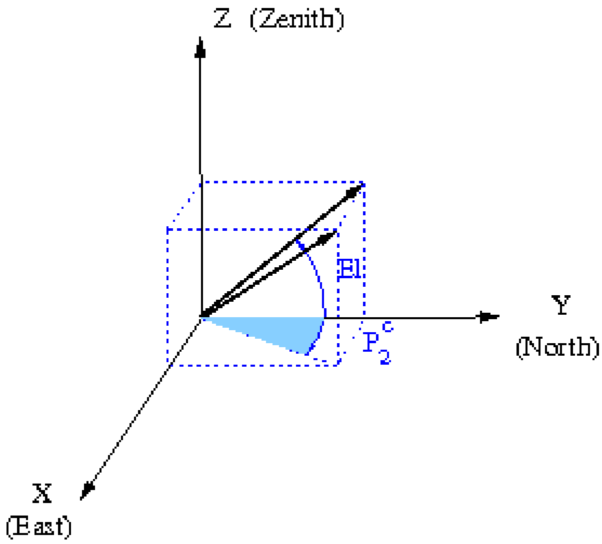

Figura 3: *Sketch of the collimation error. The antenna is pointing towards azimuth 0 and elevation El. The feed has a small pointing error and points towards the same elevation but a different azimuth (P_2^c).*

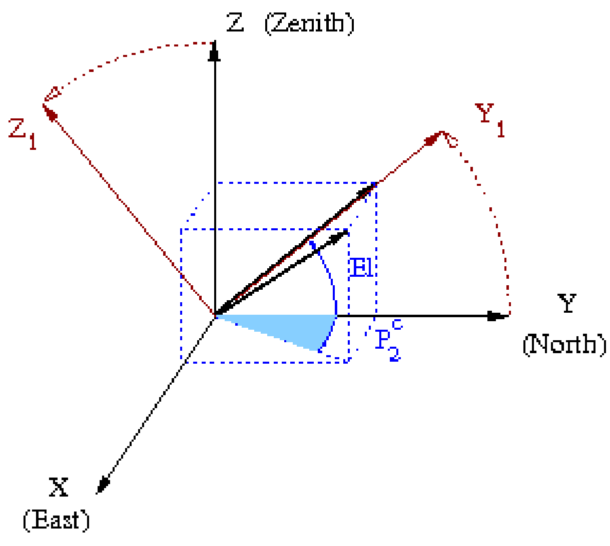

Figura 4: *Rotation of an angle El around the X axis*

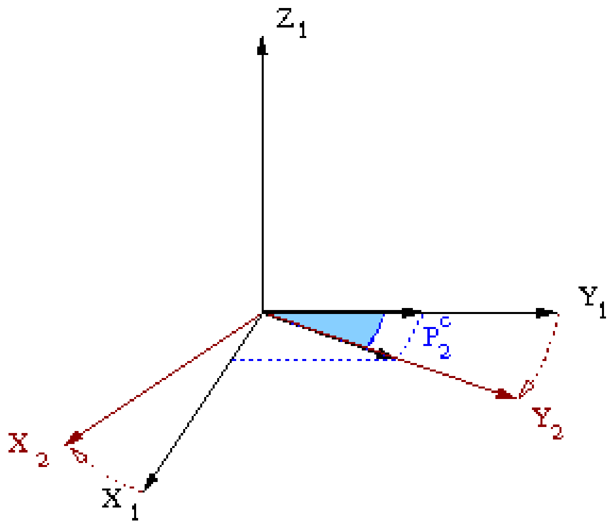

Figura 5: *Rotation of an angle P_2^c around the Z_2 axis*

where

$$\begin{pmatrix} i_3 \\ j_3 \\ k_3 \end{pmatrix} = \begin{pmatrix} i' \\ j' \\ k' \end{pmatrix} = \begin{pmatrix} \cos El' \sin Az' \\ \cos El' \cos Az' \\ \sin El' \end{pmatrix}$$

To obtain the error in elevation we examine k':

$$\begin{aligned} k' &= \sin El' \\ &= \sin El \sin P_2^c\, i + \sin El \cos El(\cos P_2^c - 1)\, j + (\sin^2 El \cos P_2^c + \cos^2 El)\, k \end{aligned}$$

replacing i, j and k by its values from equation 9 we get:

$$\begin{aligned} \sin El' = {} & \sin El \sin P_2^c \cos El \sin Az + \sin El \cos El(\cos P_2^c - 1) \cos El \cos Az + \\ & (\sin^2 El \cos P_2^c + \cos^2 El) \sin El \end{aligned}$$

taking into account that $Az = 0$,

$$\sin El' = \sin El \cos^2 El(\cos P_2^c - 1) + (\sin^2 El \cos P_2^c + \cos^2 El) \sin El$$

since P_2^c is little we may approximate $\cos P_2^c \simeq 1$ and we get:

$$\begin{aligned} \sin El' &\simeq (\sin^2 El + \cos^2 El) \sin El \\ &\simeq \sin El \end{aligned}$$

and that means that

$$\Delta El = El - El' = 0 \tag{17}$$

This result is obvious since we are only taking into account here the collimation error in a plane paralell to the ground.

To obtain the error in azimuth let us examine i':

$$\begin{aligned} i' &= \cos El' \sin Az' \\ &= \cos P_2^c i - \sin P_2^c \cos El j - \sin El \sin P_2^c k \\ &= \cos P_2^c \cos El \sin Az - \sin P_2^c \cos^2 El \cos Az - \sin^2 El \sin P_2^c \end{aligned}$$

since $Az = 0$,

$$\begin{aligned} \cos El' \sin Az' &= -\sin P_2^c \cos^2 El - \sin^2 El \sin P_2^c \\ &= -\sin P_2^c \end{aligned}$$

Since $Az' = Az - \Delta Az$, and $El = El'$:

$$\cos El \sin(Az - \Delta Az) = -\cos El \sin \Delta Az = -\sin P_2^c$$

P_2^c is expected to be small and the previous expression may be rewritten as:

$$\cos El \sin \Delta Az \simeq P_2^c \tag{18}$$

And the error in azimuth is, for small ΔAz,

$$\Delta Az \simeq P_2^c \sec El \tag{19}$$

The previous expression means that when the antenna points towards the horizon (elevation is cero) the azimuth error is P_2^c, and it increases as the elevation increases being ∞ when elevation is $90°$. Figure 6 shows this dependence. This behaviour has a simple physical meaning.

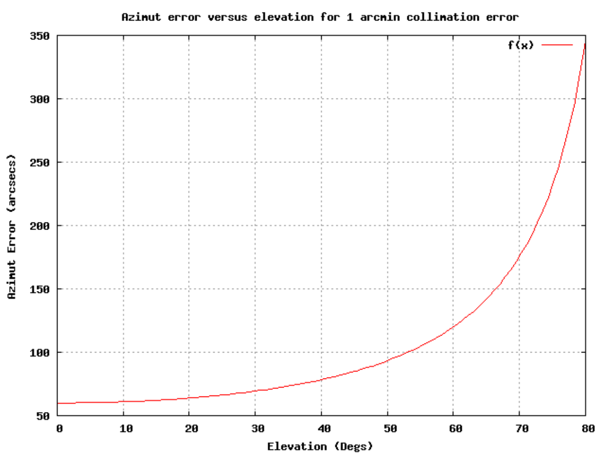

Figura 6: *Azimuth error as a function of elevation. P_2 has been chosen to be 1 arcmin*

Figure 7 shows a cenital view of a semisphere, and a large value for P_2^c to help the reader. The circles represent the azimuth movement for different elevations. Elevation cero gives a larger azimuth track than for example, elevation $60°$. Let us assume that on the horizon the error is P_2^c. As we move in elevation the projection of the pointing direction draws a straight line, parallel to the Y axis. The azimuth for each elevation is the angle that has to be drawn along each elevation circle to arrive to the pointing direction. When the azimuth is greater than $90°$ the azimuth error has no physical meaning, since the circles for elevations greater than the one that is tangent to the projection pointing line, do not cross it. This is reasonable since equation 19 only remains valid for small P_2^c angles.

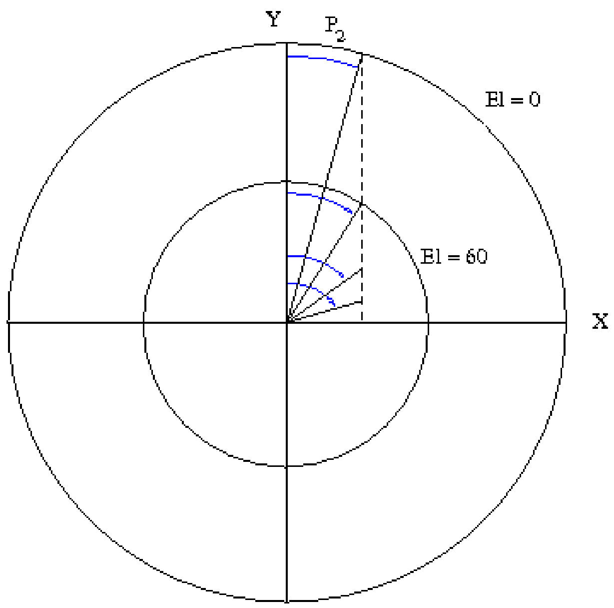

Figura 7: *Semisphere viewed from above. Concentric circles represent different azimuth movement for a given elevation. Azimuth is the angle between axis Y and the position line. A collimation error is seen as a vertical line parallel to the Y axis. The azimuth error increases as elevation increases.*

5.4. Lack of orthogonality between the azimuth and elevation axis. P_3^o

This error appears when the azimuth axis and the elevation axis are not orthogonal. The physical meaning for this error may be seen with the aid of Figure 8. Given an azimuth, for example $Az = 0$, the elevation axis is in plane XZ but makes an angle $-P_3^o$ with axis X. When the elevation changes, the pointing vector will travel on a plane which passes through axis Y and forms an angle $-P_3^o$ with plane YZ. The projection of the pointing vector on plane XY generates a curve on that plane which depends on elevation. The angle of each point of the curve with the Y axis is the azimuth error. The azimuth error is independent of the azimuth.

Without loss of generality we will assume the antenna points towards azimuth North ($Az = 0$). We can obtain a new coordinate system by rotating the old system an angle P_3^o counter

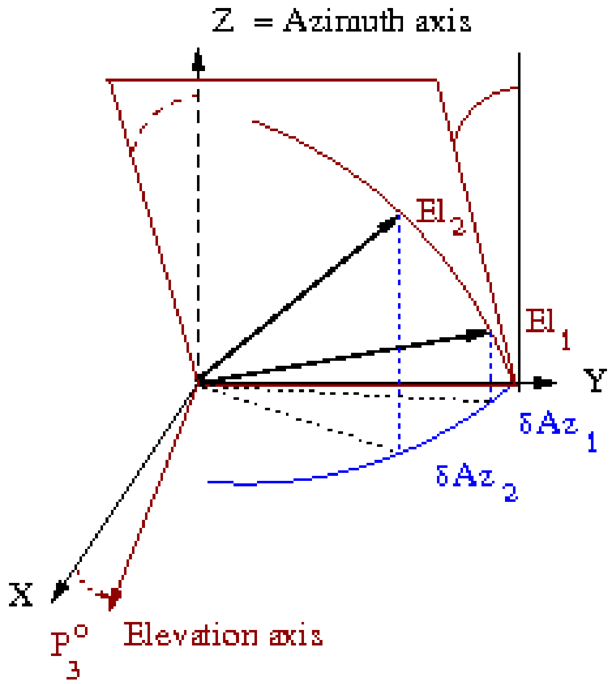

Figura 8: *Lack of orthogonality between the azimuth and the elevation axis. The elevation axis is in the XZ plane but makes an angle $-P_3^o$ with X axis. When the antenna moves in elevation the pointing vector travels on a plane which forms a $-P_3^o$ angle with plane YZ.*

clockwise around the Y axis. Using equation 14,

$$\vec{p'} = \begin{pmatrix} i' \\ j' \\ k' \end{pmatrix} = \begin{pmatrix} \cos P_3^o & 0 & -\sin P_3^o \\ 0 & 1 & 0 \\ \sin P_3^o & 0 & \cos P_3^o \end{pmatrix} \begin{pmatrix} i \\ j \\ k \end{pmatrix}$$

To obtain the error in elevation we can examine k',

$$k' = \sin El' = \sin P_3^o\, i + \cos P_3^o\, k = \sin P_3^o \cos El \sin Az + \cos P_3^o \sin El$$

since $Az = 0$,

$$\sin El' = \cos P_3^o \sin El$$

and as P_3^o is small:

$$\sin El \simeq \sin El'$$

and there is no error in elevation:

$$\Delta El \simeq 0 \tag{20}$$

To obtain the error in azimuth we examine i',

$$\begin{aligned} i' &= \cos El' \sin Az' \\ &= \cos P_3^o\, i - \sin P_3^o\, k \\ &= \cos P_3^o \cos El \sin Az - \sin P_3^o \sin El \end{aligned}$$

Since $Az = 0$, $El = El'$ and $Az' = Az - \Delta Az$ the previous expression is equivalent to,

$$-\cos El \sin Az' = -\sin P_3^o \sin El$$

and since P_3^o is small we may approximate the previous equation to,

$$\sin \Delta Az \simeq P_3^o \tan El \tag{21}$$

which for small values of ΔAz,

$$\Delta Az \simeq P_3^o \tan El \tag{22}$$

If the angle between the azimuth axis and the elevation one is smaller than $90°$, P_3^o is positive and the azimuth error is positive, as shown in Figure 8. According to 22 the azimuth error increases with elevation. For example at elevation $90°$ the pointing vector lays on plane XZ and forms an angle P_3^o with axis Z and the azimuth error is so large that it has no meaning.

Expresion 22 can also be obtained by a careful inspection of Figure 8. The tangent of the angle for each point of the curve on the XY plane can be obtained by dividing the side opposite to the angle by the side close to the angle. X coordinate is $\cos El$. Y coordinate is $\sin El$ projected onto the X axis, that is $\sin El \sin P_3^o$. Therefore:

$$\tan(\Delta Az) = \sin P_3^o \sin El / \cos El \tag{23}$$

which for small ΔAz values is the same expression as 22.

Figure 9 shows the dependence of the Az error with elevation. This dependence is only valid for small P_3^o errors and, therefore, the previous figure shows the case when $P_3^o = 1'$. For low elevations this error is small, it is equivalent to P_3^o at $45°$ and increases dramatically beyond $45°$.

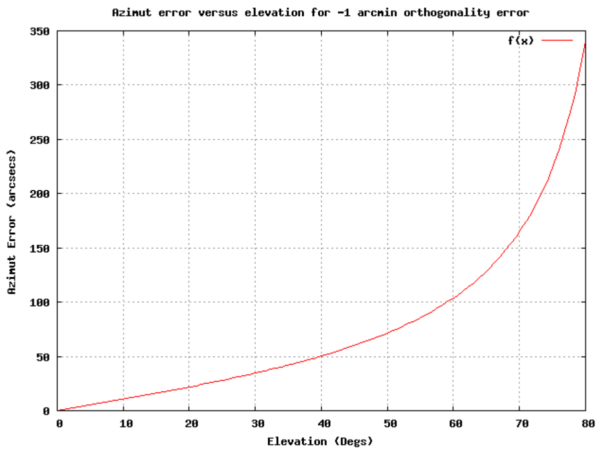

Figura 9: *Azimuth error as a function of elevation for an error of* -1 *arcmin in the orthogonality of both axis.*

5.5. Tilt of the azimuth axis

The lack of perpendicularity of the azimuth axis with respect to the ground plane may be decomposed in two directions, along the East - West direction, and along the North - South direction. Each case will be treated separately in the following two sections.

5.5.1. Tilt along the E-W direction. $\mathbf{P_4^e}$

Figure 10 shows the case in which the azimuth axis is tilted an angle P_4 towards the East. This situation is very similar to the one depicted on Figure 8 except for one difference, azimuth axis and elevation axis remain perpendicular and therefore azimuth axis is not aligned with Z axis.

We can obtain a new coordinate system by rotating the old system an angle P_4^e counter-clockwise around the Y axis. Using equation 14,

$$\vec{p'} = \begin{pmatrix} i' \\ j' \\ k' \end{pmatrix} = \begin{pmatrix} \cos P_4^e & 0 & -\sin P_4^e \\ 0 & 1 & 0 \\ \sin P_4^e & 0 & \cos P_4^e \end{pmatrix} \begin{pmatrix} i \\ j \\ k \end{pmatrix}$$

To obtain the error in elevation we can examine k',

$$k' = \sin El' = \sin P_4^e \, i + \cos P_4^e \, k = \sin P_4^e \cos El \sin Az + \cos P_4^e \sin El$$

Since P_4^e is small we will approximate $\cos P_4^e = 1$ and $\sin P_4^e = P_4^e$. Then:

$$\sin El' = P_4^e \cos El \sin Az + \sin El$$

Taking into account that:

$$\sin El' - \sin El = 2 \cos \frac{(El' + El)}{2} \sin \frac{(El' - El)}{2}$$

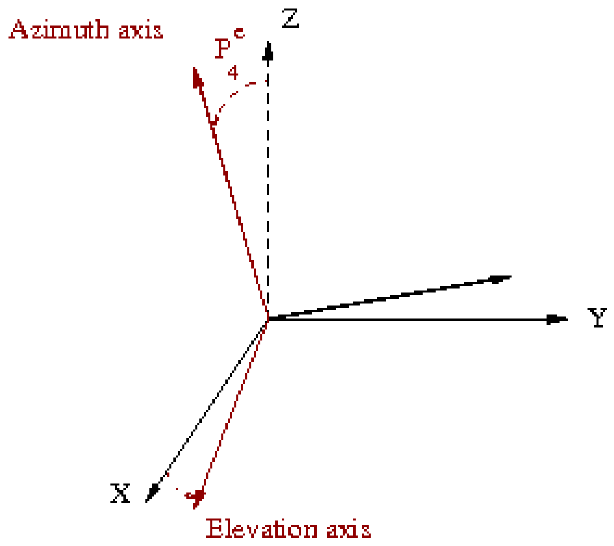

Figura 10: *Tilt of the azimuth axis towards the east by a quantity* P_4^e.

and

$$El' + El = 2El + \Delta El$$

we obtain that:

$$\begin{aligned} P_4^e \cos El \sin Az &= 2\cos(El + \frac{\Delta El}{2})\sin(\frac{\Delta El}{2}) \\ &= 2(\cos El \cos(\frac{\Delta El}{2}) - \sin El \sin(\frac{\Delta El}{2}))\sin(\frac{\Delta El}{2}) \\ &\simeq 2(\cos El - \frac{\Delta El}{2}\sin El)\frac{\Delta El}{2} \\ &\simeq \Delta El \cos El - \sin El \frac{\Delta El^2}{2} \end{aligned}$$

where we have approximated $\cos(\Delta El/2) \simeq 1$ and $\sin(\Delta El/2) \simeq \Delta El/2$ because ΔEl is small.

For small elevation errors one can consider that:

$$\cos El >> \sin El \frac{\Delta El}{2}$$

We have tested the previous comparison with $1'$ elevation error and 89.5 degrees elevation, which we believe is one of the worst cases, and the inequality holds true:

$$\cos 89^\circ = 0{,}017 >> \sin 89^\circ \frac{1'}{2} = 4{,}63\,10^{-5}$$

Obviously the approximatino does not hold when elevation is 90 degrees or the elevation error is larger 3 degrees. Therefore we will accept that:

$$P_4^e \cos El \sin Az \simeq \Delta El \cos El$$

Finally we obtain:

$$\Delta El = P_4^e \sin Az \tag{24}$$

Expression 24 means that when the antenna points towards the North or the South the error is 0, while the error is maximum when pointing towards the East and the West. In between the error varies following the sine function. This periodic tilt is summarized in Figure 11.

To obtain the error in azimuth we examine i',

$$\begin{aligned} i' &= \cos El' \sin Az' \\ &= \cos P_4^e\, i - \sin P_4^e\, k \end{aligned}$$

Hence,

$$\begin{aligned} \cos El' \sin Az' &= \cos P_4^e \cos El \sin Az - \sin P_4^e \sin El \\ &\simeq \cos El \sin Az - P_4^e \sin El \end{aligned}$$

where, since P_4^e is small, we have approximated $\cos P_4^e \simeq 1$ and $\sin P_4^e \simeq P_4^e$. Expanding El' and Az':

$$\begin{aligned} \cos El \sin Az - P_4^e \sin El &= \cos El' \sin Az' \\ &= \cos(El + \Delta El) \sin(Az + \Delta Az) \\ &= (\cos El \cos \Delta El - \sin El \sin \Delta El)(\sin Az \cos \Delta Az + \cos Az \sin \Delta Az) \\ &\simeq (\cos El - \sin El \Delta El)(\sin Az + \Delta Az \cos Az) \end{aligned}$$

where we assumed that $\cos \Delta Az \simeq 1$, $\cos \Delta El \simeq 1$, $\sin \Delta Az \simeq \Delta Az$ and $\sin \Delta El \simeq \Delta El$. Using expression 24,

$$\begin{aligned} \cos El \sin Az - P_4^e \sin El &= (\cos El - \sin El P_4^e \sin Az)(\sin Az + \Delta Az \cos Az) \\ &= \cos El \sin Az - \sin El P_4^e \sin^2 Az - \Delta Az \cos Az \sin Az \sin El P_4^e + \\ &\quad \Delta Az \cos Az \cos El \end{aligned}$$

Grouping terms:

$$P_4^e \sin El(\sin^2 Az - 1) = \Delta Az \cos Az(\cos El - P_4^e \sin Az \sin El)$$

and:

$$\begin{aligned} \Delta Az &= -\frac{P_4^e \sin El \cos^2 Az}{\sin El \cos Az(ctgEl - P_4^e \sin Az)} \\ &= -\frac{P_4^e \cos Az \tan El}{1 - P_4^e \sin Az\, tanEl} \end{aligned}$$

and finally:

$$\Delta Az \simeq P_4^e \tan El \cos Az \tag{25}$$

where we have considered that $1 >> -P_4^e \sin Az \tan El$. This approximation should be taken carefully and as previouly, we have tested it with $1'$ elevation error and 89 degrees elevation and the aproximation is valid. Obviously if the error is very large or elevation is 90 degrees the above does not hold.

The azimuth error described in 25 reduces to 22 when azimuth is 0. In all other cases, where Az is not 0, the error is obtained by multiplying that same expression by $\cos Az$.

Taking into account the approximations performed in this section, expresions 24 and 25 hold for very high elevations only if the error is of the order of a few arcmins at most. For elevations smaller than 88 degrees the previous expresions are valid for higher errors.

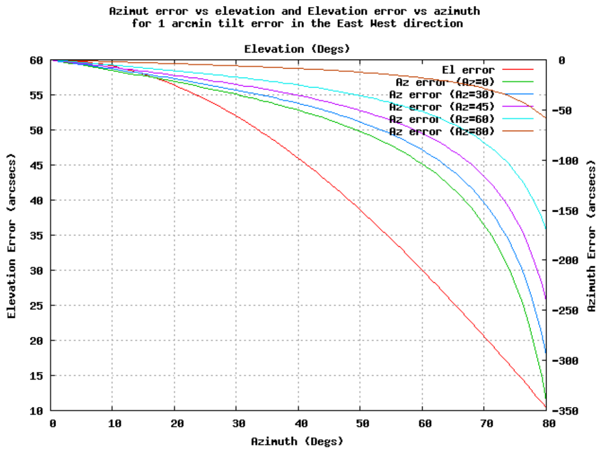

Figura 11: *Red curve represents the elevation error as a function of azimuth for an error of* 1 *arcmin in the tilt of azimuth axis. The other curves represent the azimuth error as a function of elevation (see top and right axis) and for different azimuth angles.*

5.5.2. Tilt of the azimuth axis along the N-S direction

This effect has the same treatement as the tilt of the azimuth axis along the E-W direction. Since the E-W axis forms an angle of $90°$ with the N-S axis, expressions 24 and 25 can be reused by replacing Az by $90° - Az$. Therefore, assuming P_5^n is the tilt of the azimuth axis towards the North:

$$\Delta El = P_5^n \cos Az$$
$$\Delta Az \simeq -P_5^n \sin Az \tan El$$

These expressions make sense since one should get the same kind of error for azimuth and elevation but expects the same periodic dependence of the errors on azimuth shifted 90 degrees.

5.6. Gravitational bending / homologous deformation

Naively one could consider that the main reflector and the secondary reflector are perfect rigid structures connected by a bending rod. The rod can be considered a cantilevered beam, that is, the position and slope of the reflector end of the rod is fixed. The tilt angle of the rod is proportional to the torque exerted at the secondary reflector end:

$$\begin{aligned}\Delta El &= k\tau = k\vec{r}\times\vec{F} = k|r||F|\sin\theta = k|r||F|\sin(90+El)\\ &= k_0\cos El\end{aligned}$$

This simplistic derivation could be applied to all elements of the structure which may be considered to be formed by interconnected rods forming different angles with the horizontal. In that case the elevation error can be considered a linear combination of sines and cosines.

Gravity causes errors only in elevation since the telescope structure is subject to torques that vary with the elevation of the antenna. Since the gravity is vertical and the antenna is a co-rotating structure around a vertical axis these errors do not depend on azimuth. Therefore errors in elevation are a given function of elevation which is periodic under a rotation of $360°$. Azimuth errors should be cero as long as we consider the antenna to have a simmetric structure with respect a vertical plane that contains the mechanical axis of the paraboloid.

According to Von Hoerner & Wong (1975) the displacement of joints for a given tilt in a co-rotating system fixed in the structure is:

$$\vec{\delta}\phi = \vec{a}\cos\phi + \vec{b}\sin\phi = \vec{c}\cos(\phi-\Omega)$$

where δ is a displacement vector of the points of the joints and a and b are displacement vectors at two orthogonal positions. These displacement vectors depend on the gravity vector and the stiffness matrix. Further development of these equations is out of the scope of this report and involves one of the methods of structural analysis, the displacement method or matrix stiffness method, which is the most common implementation of the finite element method. However the physical interpretation of the previous equation is that the deformations of a structure rotated $360°$ under the effect of gravity are described by a simple cosine wave with some amplitude c and a phase shift (Ω).

The previous equation can be rewritten so that it matches our notation. The global vector of displacement can be considered a pointing error in elevation for the whole structure and the amplitudes of the sine and cosine functions, scalar quantities:

$$\Delta El = P_8\cos El + P_9\sin El$$

Figure 12 shows 4 examples where $|P_8| = |P_9| = 1'$ but may have different signs. The behaviour is clearly different in each case.

6. Effects on Nasmyth telescopes.

This section has been adapted from a private communication by A. Barcia who developed this model independently. No previous reference to work on Nasmyth telescopes was found in the bibliography.

In the previous section we analyzed the pointing errors for a Cassegrain antenna. In the classic Cassegrain antenna the feed is located close to the Cassegrain focus and moves together with the antenna when it moves in elevation. Therefore, the reflector, the electric axis and the

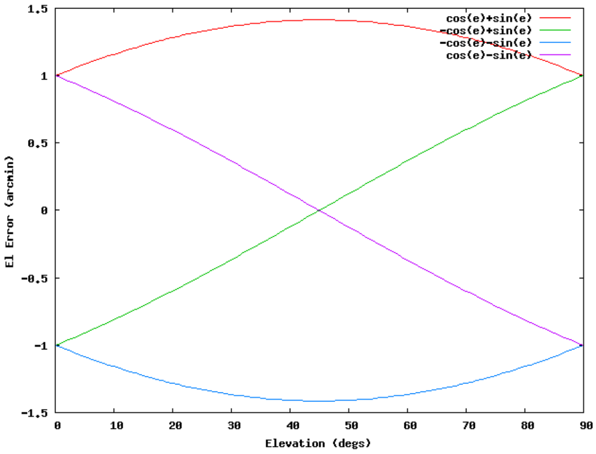

Figura 12: *Each curve represents a different combination which is specified by the title. The graphics shown are:* $\cos El + \sin El$, $\cos El - \sin El$, $-\cos El + \sin El$ *and* $-\cos El - \sin El$. *Amplitude is assumed to be 1 arcmin.*

mechanical axis move together. Some of the effects discussed in the previous section should be reviewed for the Cassegrain-Nasmyth telescopes.

In the Cassegrain-Nasmyth antenna a mirror M3 carries the image of the Cassegrain focus (F2) to the Nasmyth focus (F3), which is in the elevation axis. In order to have F3 in the elevation axis, M3 should always form an angle of 45 degrees with the mechanical axis and has to rotate around the elevation axis synchronous with the antenna in elevation. At elevation 0, M3 is perpendicular to the floor and at $90°$ elevation it forms an angle of $45°$ with the floor. Usually a fourth mirror (M4) projects F3 to another focal point F4, see Figure 13. When the antenna turns around the elevation axis the Cassegrain focus (F2) also turns, but the Nasmyth focus (F3) and F4 remain static. In some antennas the receiver is not placed in F4 since the beam is deviated towards other points (F5, F6, F7, ...) being reflected by aditional mirrors (M5, M6, ...). These ones do not need to be taken into account now. While the image of F4 (ie. F3) according to M4 is in the elevation axis, and the image of F3 (ie. F2) according to M3 is in the mechanical axis, both axis the electrical and mechanical one will match, and there will be no pointing error.

A missalignment of M3 causes a deviation of F2 from the mechanical axis. The position of F2 remains fixed relative to the mechanical axis and therefore moves solidarious with it and the antenna, tracing an arc of circumference. This case is the same as the one with a Cassegrain antenna which has a collimation error. F2 may deviate from the mechanical axis by a missalignment of M3. When M3 rotates around the elevation axis, F2 moves solidarious with it tracing an arc of circumference and therefore moving with the whole antenna. The position of F2 relative to the mechanical axis does not change and the electrical and mechanical axis which do not coincide move together as in the Cassegrain case. ical case. M3 may be misaligned by two effects, which can happen individually or combined.:

- The center of M3 is in the intersection of the mechanical axis and the elevation axis but F3 does not form in the elevation axis due to a wrong tilt of M3.

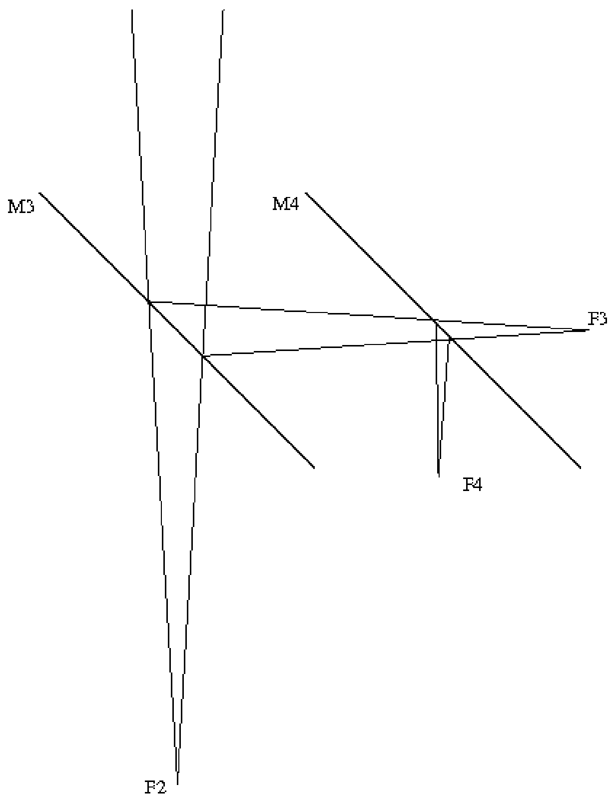

Figura 13: *Representation of the focus in a Nasmyth antenna.*

- The center of M3 is in the mechanical axis but not in the elevation axis.

In both cases F3 is formed out of the elevation axis and when M3 moves solidarious with the antenna F3 traces a circumference around the elevation axis. The projected image

Therefore the missalignement of Nasmyth mirror M3 causes the following errors:

$$\delta El = P_7^{m3}$$
$$\delta Az = P_2^{m3}$$

If M3 is correctly aligned but the receiver is not, or any of the mirrors M4, M5, M6, ... are not aligned, the image F3 of the receiver will not be formed in the elevation axis. When the antenna (ie. M3, M2, M1 and the mechanical axis) turns in elevation, F3 remains static and its position relative to the antenna changes. The effect of the turn on the relative position of the electrical and mechanical axis is the same as if the antenna remains static and F3 turns, in the other sense, tracing a circumference around the elevation axis. Its image, F2, will describe a circumference of the same radius around the mechanical axis and the electrical axis will trace a cone with a vertex coincident with the vertex of the equivalent paraboloid around the mechanical axis. Hence the error in azimuth and elevation will be:

$$\begin{aligned}
\delta El &= \Delta_s \cos(El + K) \\
&= \Delta_s K_1 \cos El - \Delta_s K_2 sinEl \\
&= P_9^{m4} \cos El + P_8^{m4} \sin El \\
\delta Az &= \Delta_s \sin(El + K) \sec El \\
&= \Delta_s K_1 \tan El + \Delta_s K_2 \\
&= P_3^{m4} \tan El + P_1^{m4}
\end{aligned}$$

Where Δ_s is the total angular error in the location of the receiver. K is a constant that depends on the relative orientation of the mirrors relative to the elevation axis and on the positioning error in the X and Y axis. If all mirrors form an angler of 45 degrees with the ideal axis, K is proportional to the p_x/p_y ratio or to p_y/p_x ratio (depending on the used mirrors) where p_x and p_y are the errors in the fixed reference system where y axis is perpendicular to the floor, x parallel to it and z is along the ray direction.

The previous expressions add to the expression obtained for Cassegrain reflectors.

7. Final algorithm

Summing up all the previous terms we get the following expressions for a **Cassegrain** for the azimuth error:

$$\begin{aligned}
\Delta Az &= P_1^{enc} + P_2^c \sec El + P_3^o \tan El - P_4^e \cos Az \tan El - P_5^n \sin Az \tan El \\
&= P_1 + P_2 \sec El + P_3 \tan El - P_4 \cos Az \tan El + P_5 \sin Az \tan El \qquad (26)
\end{aligned}$$

and the elevation error:

$$\begin{aligned}
\Delta El &= (P_7^{enc} + P_7^c) + P_4^e \sin Az + P_5^n \cos Az + P_8^g \sin El + P_9^g \cos El \\
&= P_7 + P_4 \sin Az + P_5 \cos Az + P_8 \sin El + P_9 \cos El \qquad (27)
\end{aligned}$$

The discrepancy in the signs of each coefficient when compared with equations 3, 4, 7 and 8 arises from the sign of the error angle. In previous sections this sign has been explained. Table 2 summarizes the cause of each error and the sign used.

Parameter	Physical meaning
P_1	Azimuth encoder offset. If positive the antenna always points towards larger azimuth values.
P_2	Collimation error. If positive the feed points towards larger azimuth values.
P_3	Lack of orthogonality between the azimuth and elevation axis. If positive both axis form an angle smaller than $90°$.
P_4	Tilt of azimuth axis along a E-W direction. If positive the axis is tilted towards the East.
P_5	Tilt of azimuth axis along a N-S direction. If positive the axis is tilted towards the South.
P_7	Elevation encoder offset. If positive the antenna always points towards higher elevations.
P_8	Gravitational effects. Combines with P_9 and has no simple meaning
P_9	Gravitational effects. Combines with P_8 and has no simple meaning

Cuadro 2: *Explanation of the pointing coefficients for a Cassegrain whose pointing errors are described by 26 and 27.*

To correct for these pointing errors in the 40 m telescope it is necessary to feed the Antenna Control Unit with the opposite signs obtained here. For example if the constant pointing error in elevation, P_7, is positive, the telescope is pointing "too high" and the ACU needs to get a "$-P_7$" value.

For a **Nasmyth** antenna the algorithms are finally the same but the explanation for the parameters is different. Azimuth error is:

$$\begin{aligned}\Delta Az &= P_1^{enc} + P_1^{m4} + P_2^{m3}\sec El + (P_3^{m4} + P_3^{o})\tan El - P_4^{e}\cos Az\tan El - P_5^{n}\sin Az\tan El \\ &= P_1 + P_2\sec El + P_3\tan El - P_4\cos Az\tan El + P_5\sin Az\tan El \end{aligned} \quad (28)$$

and the elevation error:

$$\begin{aligned}\Delta El &= (P_7^{enc} + P_7^{c}) + P_4^{e}\sin Az + P_5^{n}\cos Az + (P_8^{g} + P_8^{m4})\sin El + (P_9^{g} + P_9^{m4})\cos El \\ &= P_7 + P_4\sin Az - P_5\cos Az + P_8\sin El + P_9\cos El \end{aligned} \quad (29)$$

See table 3 for a summary of coefficients.

7.1. Determination of parameters

The determination of the unknown coefficients (P_1 to P_9) is done performing pointing measurements which cover most part of the sky. A good coverage of the sky offers more guarantees

Parameter	Physical meaning
P_1	Azimuth encoder offset. If positive the antenna always points towards larger azimuth values. This term also includes positioning errors for receivers in the Nasmyth focus.
P_2	Collimation error. It includes positioning errors for Nasmyth mirrors
P_3	Lack of orthogonality between the azimuth and elevation axis. If positive both axis form an angle smaller than $90°$. This term also includes positioning errors for receivers in the Nasmyth focus.
P_4	Tilt of azimuth axis along a E-W direction. If positive the axis is tilted towards the East
P_5	Tilt of azimuth axis along a N-S direction. If positive the axis is tilted towards the South
P_4	Tilt of azimuth axis along a E-W direction and N-S direction. Since it is a combination, the sign does not inform about the direction of the axis tilt.
P_5	Tilt of azimuth axis along a N-S direction and E-W direction. Since it is a combination, the sign does not inform about the direction of the axis tilt.
P_7	Elevation encoder offset. If positive the antenna always points towards higher elevations. It also includes positioning errors for Nasmyth mirrors.
P_8	Gravitational effects. It also includes positioning errors for receivers in the Nasmyth focus.
P_9	Gravitational effects. It also includes positioning errors for receivers in the Nasmyth focus.

Cuadro 3: *Explanation of the pointing coefficients for a Nasmyth whose pointing errors are described by 26 and 27.*

to discover the dependencies with elevation and azimuth. Since there are 9 unknown variables and the system is overdetermined the solution is obtained by least square analysis. Let us assume that the antenna performed n pointing observations, each with one azimuth stroke and one elevation stroke. The errors obtained by radio pointing strokes or directly measured using an optical CCD are angles in the sky and therefore have to be transformed to azimuth errors. That means that errors in the horizontal axis need to be divided by $\cos El$, where El is the elevation at which the measurements are done:

$$\delta Az = \delta X \sec El$$

where δX is the error angle in the sky along the horizontal axis, and could also be called "collimation" error.

The system of equations to solve would then be:

$$\begin{pmatrix} . \\ 2n \text{ elements} \\ . \end{pmatrix} = \begin{pmatrix} 1......,9 \\ ..... \\ 2n \text{ elements} \\ ..... \\ 1......,9 \end{pmatrix} \begin{pmatrix} P_1 \\ ... \\ P_9 \end{pmatrix}$$

This system can be rewritten:

$$E = AP$$

where E is a matrix of $1 \times 2n$, A is a matrix of $2n \times 9$ dimensions and P is a matrix of 1×9. This system is solved by obtaining :

$$\text{minimize}|\text{E} - \text{AP}|$$

The author of this report developed a C++ aplication which used LAPACK to solve the system and Qt to represent the errors as a function of azimuth and elevation. LAPACK provides several functions to solve the problem: xGELSX, xGELSY, xGELSS, and xGELSD. The x preceding the names of these functions are for using ral and complex numbers with single or double precision.

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